CALCULUS I, SPRING 2013

PROJECT #1 - LIMITS AND AN OIL SPILL

1. Introduction

In this project, you will explore the theoretical definition of a limit at infinity and then apply that to a hypothetical situation of an oil spill. You can work in groups of up to three people (of your own choosing) or work by yourself. If you are working in a group, **each member of the group must write the solution for at least two of the problems** to help guarantee that all members of the group are pulling their weight. There is no individual score, however- only one answer sheet should be turned in and everyone in the group shares the same grade.

Timeline: Questions 1, 2 and 3 will be due in one week (Wednesday, February 6) and the remaining part of the project will be due on Wednesday, February 13, including corrections from any part of questions 1-3 that were wrong. Note that the grading is based both on the correctness of your answers as well as your method of getting the answer. It is very important that you show your work in order to receive full credit.

Scoring

- First part: 20% of total grade
- Second part: 70% of total grade
- Corrections to first part: 10% of total grade (if the first part is totally correct, you get this 10% for free).

2. Oil Spills and Dispersant Chemicals

On April 20, 2010, the Deepwater Horizon drilling rig exploded and initiated the worst marine oil spill in recent history. Oil gushed from the well for three months and released millions of gallons of crude oil into the Gulf of Mexico. One technique used to help clean up during and after the spill was the use of the chemical dispersant Corexit. Oil dispersants allow the oil particles to spread more freely in the water, thus allowing the oil to biodegrade more quickly. Their use is debated, however, because some of their ingredients are carcinogens and the use of oil dispersants can increase toxic hydrocarbon levels in sea life. Over time, the pollution caused by the oil spill and the dispersants will eventually diminish and the sea life will return, more or less, to its previous condition. In the short term, however, pollution can raise serious issues in regards to the health of the local sea life and the safety of fish and shellfish for human consumption. We now look at what the impact would be on a hypothetical oil spill in the Puget Sound off the coast of Washington State.

3. Oil Spill Case Study

Due to a thick fog, an oil tanker gets lost on its way through the San Juan Islands in northwestern Washington State. The tanker crashes near Orcas Island and leaks oil into East Sound, a large bay on the interior of the island. Because the volume of oil is large and there is little water flow into the sound, the oil company uses a large amount of oil dispersants to help fix the problem, but the marine ecosystem is damaged nonetheless. The ensuing legal case ends up in arbitration, and the arbitrator brings you in to help with some of the mathematics presented in the debate. The oil company argues that the water quality in the sound will eventually return to the level it was before the spill, so they claim that no monetary restitution is necessary. The local fishermen, however, claim that even if the water quality returns to its normal level, it will still be many years before the seafood caught in the sound is safe to eat, and they should be compensated for the damage done to the seafood industry in the meantime. Both sides have presented their case to the arbitrator, and the arbitration is in recess until a decision is made.

Certain facts are not in dispute:

- Immediately after the use of oil dispersants ended, the water in the sound had an average of 1000 parts per million (ppm) of dangerous pollutants.
- A safe level for fish is considered to be 500 ppm, and a safe level for shellfish is 100 ppm.
- The normal level of pollutants in the water, before the oil spill, is estimated to be 1 ppm.
- The fishing industry in East Sound earns approximately $80,000 in profits a year.
- Due to the fact that no major rivers flow into the sound, the water is mainly flushed out during tidal changes. Thus it is estimated that only 25% of the water in the sound is flushed out each year.
- The amount of pollutants left in the water can be modeled by the formula

  $$p(t) = A(1-c)^t + 1 \text{ ppm}$$

  where A is the original amount of pollutants in the water, c is the percent change in the water written as a decimal, and t is time in years.

4. Questions

Using the facts given above, answer the following questions to help the arbitrator make an informed decision.

(1) With $A = 1000$ ppm and $c = 0.25$, use a table of values to estimate $\lim_{t\to\infty} p(t)$. Does that limit confirm or refute the oil company's claim that the water quality will return to its original level? Explain.
(2) (a) Find the amount of time in years it will take for the level of pollutants to go below 500 ppm. If you are having trouble solving the equation, look at the first part of the Addenda for how to solve an exponential equation.
    (b) Find the amount of time in years it will take for the level of pollutants to go below 100 ppm.
    (c) Do the calculations in parts (a) and (b) confirm or refute the claims made by the fishermen? Explain.
(3) How many years, according to the model, will it be before the water quality returns to the level it was before the oil spill? Explain.
(4) What would you recommend to the arbitrator in terms of how much, if any, restitution should be given to the fishermen?
(5) Suppose that when the arbitration reconvenes, new facts indicate that water is actually being flushed out of East Sound at a rate of 40% a year, and the allowable level of pollutants in shellfish has dropped to 50 ppm. Does this change your conclusion, and if so, how?
(6) Use the ϵ-definition of a limit at infinity to prove that your calculation of $\lim_{t\to\infty} p(t)$ from part (1) is correct. In other words, you need to find an inequality that gives me an appropriate value of N for any value of ϵ that I choose. See the second part of the Addenda to find the ϵ-definition of a limit at infinity and how to use it to prove that a function has a limit.
(7) Use the ϵ-definition of a limit at infinity to prove that

$$\lim_{t\to\infty} A(1-c)^t + 1 = 1$$

for any positive value of A and any $0 < c < 1$. Make sure to explain your reasoning, in particular you need to explain when and why the inequality changes direction.

For more information on the continuing effects of the Deepwater Horizon oil spill, see

`http://www.floridaoilspilllaw.com`

and

`http://www.bp.com/sectiongenericarticle800.do?categoryId=9036585&contentId=7067606`

5. Addenda

5.1. **Solving an exponential equation.** To solve an equation that has a variable in the exponent, you have to use logarithms and logarithm rules. I will use the natural logarithm in an example below to show how this is done:

Example 1. Solve the equation $43 = 5 \cdot (2^x)$.

Ans: To solve the equation, I first divide the 5 to the other side, so that I have the 2^x by itself on one side of the equation. I then take the ln of both sides. Once I have $\ln 2^x$, I use logarithm properties to rewrite it as $x \cdot \ln 2$, and I can divide both sides by $\ln 2$ to get my answer for x.

$$\begin{aligned} 43 &= 5 \cdot (2^x) \\ \tfrac{43}{5} &= 2^x \\ \ln(43/5) &= \ln(2^x) \\ \ln(43/5) &= x(\ln 2) \\ \tfrac{\ln(43/5)}{\ln 2} &= x \\ x &= 3.104 \end{aligned}$$

5.2. **The ϵ-definition of limits at infinity.**

Definition 2. The limit $\lim_{x\to\infty} f(x) = L$ means that for all $\epsilon > 0$, there exists a number $N > 0$ such that $|f(N) - L| < \epsilon$.

Note: The symbol ϵ is pronounced "epsilon". In words, what the definition means is this: "If you want to tell me that the limit as x goes to infinity of $f(x)$ is equal to L, then if I give you a really small number ϵ, you have to be able to give me a large number N such that $f(N)$ is within ϵ distance of L." Here is a specific example:

Example 3. Use the ϵ-definition to prove that $\lim_{x\to\infty} \frac{x+1}{x-1} = 1$.

Ans: First, let's do a concrete example. Suppose I challenged you to find a value of N such that $f(N)$ was less than 0.001 away from 1. This means that you need to find a number N such that

$$\frac{N+1}{N-1} < 1.001$$

If you multiply both sides of the equation by $(N - 1)$, then solve the inequality, you end up with $\frac{2.001}{0.001} < N$, or $N > 2001$. This means that if I give you any number greater than 2001 and you plug it into $f(x)$, your answer will be between 1.001 and 1. We can check this by calculating $f(2025) = \frac{2026}{2024} = 1.0009881$. Note: this is basically what you are doing to answer questions 2(a) and 2(b).

For the proof, however, we cannot use a specific ϵ such as 0.001, we have to use the variable ϵ. If we have already done a concrete example, though, this is pretty easy because the steps are basically the same. What I need for the proof is a general expression for N, in terms of ϵ, such that *no matter what ϵ I give you, you can easily calculate the appropriate N.* Thus our inequality will start as

$$\frac{N+1}{N-1} < 1 + \epsilon$$

and we have

$$\begin{array}{rcl} \frac{N+1}{N-1} & < & 1+\epsilon \\ N+1 & < & (N-1)(1+\epsilon) \\ N+1 & < & N+\epsilon N-1-\epsilon \\ 1+1+\epsilon & < & N(1+\epsilon) \\ \frac{2+\epsilon}{1+\epsilon} & < & N \end{array}$$

This proves that $\lim_{x\to\infty} \frac{x+1}{x-1} = 1$ because no matter how close to 1 you want me to get, I can find an N (using the expression we just found) such that $f(N)$ will be that close to 1.