Question (1360001)

A comet orbiting Sun

A comet of mass \(2.6 \times 10^{14}\) kg orbits Sun which has a mass \(2.0 \times 10^{30}\) kg. Use the following constants and initial conditions for the comet in your calculations.

\[
\begin{align*}
\vec{r}_i &= <8.79 \times 10^{10}, 0, 0> \text{ m} \\
\vec{v}_i &= <0.54 \times 10^{4}, 0> \text{ m/s} \\
\Delta t &= 24 \text{ h} \\
G &= 6.7 \times 10^{-11} \text{ N m}^2/\text{kg}^2
\end{align*}
\]

Assume that Sun remains stationary during the orbit (i.e. it does not “wobble”).

(a) What is the position and momentum of the comet at \(t = 30\) days.

(b) What is the period of the comet’s orbit?

(c) What is the position of the comet at aphelion (the point in its orbit when it is furthest from Sun).

Solution

(a) Use an iterative approach to answering the questions. In other words, write a program that calculates the gravitational force on the comet, its momentum, and its position in small time steps (i.e. small time intervals). The steps to doing the calculations are listed in Matter & Interactions, Chapter 3.

- Define the values of constants such as \(G\) to use in the program.
- Specify the masses, initial positions, and initial momenta of the interacting objects.
- Specify and appropriate value for \(\Delta t\), small enough that the objects don’t move very far during one update.
- Create a “loop” structure for repetitive calculations to calculate force on each object, the momentum of each object, and the clock reading.

Inside the loop, the program should do the following for each object in the simulation:

(1) Calculate the (vector) forces acting on the object and sum the forces to calculate the net force on the object.

(2) Update the momentum of the object: \(\vec{p}_f = \vec{p}_i + \vec{F}_{\text{net}}\Delta t\).

(3) Update the position of the object \(\vec{r}_f = \vec{r}_i + \vec{v}_{\text{avg}}\Delta t\), where the average velocity can be approximated as \(\vec{v}_{\text{avg}} = \frac{\vec{v}_f}{\Delta t}\). (Note: there are more accurate ways to approximate the average velocity.)

(4) Update the clock reading, \(t = t + \Delta t\).
The loop repeats, performing each of the calculations above for each time step, $\Delta t$.

You can use a condition in the while loop or an if statement to stop the loop and print the position and momentum of the comet. Or, you can simply print the position and momentum of the comet and after one orbit, stop the simulation and examine the data in order to answer the questions.

Here is an example program in Python to simulate the motion of the comet. Note that the time interval is converted to seconds.

```python
from visual import *

sun=sphere(radius=1e10, pos=(0,0,0), color=color.yellow)
comet=sphere(radius=1e10, pos=(8.79e10,0,0), color=color.white)

sun.m=2.0e30
comet.m=2.6e14
G=6.7e-11
comet.v=vector(0,5.4e4,0)
comet.p=comet.m*comet.v

t=0
dt=24.0*3600

print "time", "\t", "Position", "\t", "Velocity"

while t<=30.0*24*3600:
    rate(100)
    print t, "\t", comet.pos, "\t", comet.v, comet.p

    r=comet.pos-sun.pos
    rmag=mag(r)
    rhat=r/rmag
    F=-(G*comet.m*sun.m/rmag**2)*rhat

    comet.p=comet.p+F*dt
    comet.v = comet.p/comet.m
    comet.pos=comet.pos+comet.v*dt
    t=t+dt
```

The program will stop at $t = 30$ days, which is $(30)(24)(3600) = 2592000$ seconds. The position and velocity printed by the program are:

\[
\begin{align*}
t &= 2592000.0 \\
\vec{r} &= \langle 4.20423e + 10, 1.19421e + 11, 0 \rangle \text{ m} \\
\vec{v} &= \langle -27250.5, 35495.5, 0 \rangle \text{ m/s}
\end{align*}
\]

You can easily print the momentum or calculate it. The momentum of the comet at this instant is
\[ \vec{p} = m \vec{v} = (2.6 \times 10^{14} \text{ kg})(< -27250.5, 35495.5, 0 > \text{ m/s}) = < -7.1 \times 10^{18}, 9.2 \times 10^{18}, 0 > \text{ kg m/s} \]

Since the initial conditions and constants are given to two significant figures, the position and momentum at \( t = 30 \) days can only be reported to two significant figures even though many more significant figures are used in the calculations.

(b) To find the period of the comet, to the nearest hour, use an infinite while loop (i.e. use `while 1:` and watch the simulation. You will see that the comet travels very slowly when far from Sun. As it approaches Sun, it dramatically speeds up. Stop the simulation just after it the comet completes one period. Examine the data to see when the \( x \)-velocity changes from a positive value to a negative value. That’s the time interval during which it reached perihelion.

A sketch of the path of the comet is shown below.

![Figure 1: The comet at perihelion.](image)

The program above, with an infinite while loop, will print the following time, position, and velocity data near perihelion.

<table>
<thead>
<tr>
<th>time (s)</th>
<th>position (m)</th>
<th>velocity (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>556070400.0</td>
<td>(&lt; 8.77609e + 10, -9.51264e + 09, 0 &gt;)</td>
<td>(&lt; 3039.7, 53756.1, 0 &gt;)</td>
</tr>
<tr>
<td>556156800.0</td>
<td>(&lt; 8.78959e + 10, -4.85428e + 09, 0 &gt;)</td>
<td>(&lt; 1562.61, 53916.2, 0 &gt;)</td>
</tr>
<tr>
<td>556243200.0</td>
<td>(&lt; 8.7902e + 10, -1.88794e + 08, 0 &gt;)</td>
<td>(&lt; 70.852, 53998.6, 0 &gt;)</td>
</tr>
<tr>
<td>556329600.0</td>
<td>(&lt; 8.77787e + 10, 4.47697e + 09, 0 &gt;)</td>
<td>(&lt; -1427.51, 54001.8, 0 &gt;)</td>
</tr>
<tr>
<td>556416000.0</td>
<td>(&lt; 8.7526e + 10, 9.13613e + 09, 0 &gt;)</td>
<td>(&lt; -2924.26, 53925.5, 0 &gt;)</td>
</tr>
<tr>
<td>556502400.0</td>
<td>(&lt; 8.71449e + 10, 1.37819e + 10, 0 &gt;)</td>
<td>(&lt; -4411.17, 53770.3, 0 &gt;)</td>
</tr>
</tbody>
</table>

Table 1: Position and velocity data for the comet near perihelion.

Note that between \( t = 556243200.0 \) s and \( t = 556329600.0 \) s, the comet’s \( y \)-position changes from a negative value to a positive value, and the comet’s \( x \)-velocity changes from a positive value to a negative value. Also, the maximum \( x \)-position of the comet occurs at \( t = 556243200.0 \) s. It is during this time interval that the comet crosses the \( +x \) axis and therefore completes one period. We don’t exactly the instant when it reaches
perihelion. However, our best guess for the period, using this simulation, is $t = 556243200.0 \pm 1 \text{ h}$. Let’s convert this to years.

$$T = (556243200.0 \text{ s}) \left( \frac{1 \text{ h}}{3600 \text{ s}} \right) \left( \frac{1 \text{ day}}{24 \text{ h}} \right) = 6430 \text{ day}$$

$$= (6430 \text{ day}) \left( \frac{1 \text{ Earth year}}{365 \text{ day}} \right) = 17.6 \text{ year}$$

This comet will have a period of 17.6 y. Note that this data does not correspond to an actual, observed periodic comet. However, Halley’s comet, which you have probably heard of, has a period of 76 y, its mass is between $2.2 \times 10^{14}$ and $3.0 \times 10^{14}$ kg, its distance from Sun at perihelion is $8.77 \times 10^{10}$ m, and its perihelion speed is 54 km/s. Thus, the comet just simulated does have similar orbital parameters as Halley’s comet.

The biggest numerical error in the simulation is in the approximation $\vec{v}_{avg} = \frac{\vec{p} \Delta t}{\Delta t}$. There are various ways of improving this approximation and therefore finding a more accurate determination of the period of the comet.

(c) Aphelion occurs in this case when the comet crosses the $-x$ axis as shown below.

![Figure 2: The comet at aphelion.](image)

Examine the data to find where the $x$-velocity changes from a negative value to a positive value. During the same time interval, the $y$-position should change from a positive value to a negative value. The time interval for the program above where the comet’s $y$-position changes from positive to negative is:

<table>
<thead>
<tr>
<th>time (s)</th>
<th>position (m)</th>
<th>velocity (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>255225600.0</td>
<td>$&lt;-1.93654e+12, 5.30555e+07, 0&gt;$</td>
<td>$&lt;-746.111, -2451.05, 0&gt;$</td>
</tr>
<tr>
<td>255312000.0</td>
<td>$&lt;-1.93666e+12, -1.58716e+08, 0&gt;$</td>
<td>$&lt;-743.024, -2451.05, 0&gt;$</td>
</tr>
</tbody>
</table>

Table 2: Position and velocity data for the comet near aphelion, using the position as indicator.

Therefore, judging by the $y$-position, the comet reaches aphelion between $t = 255225600.0 \text{ s}$ and $t = 255312000.0 \text{ s}$. This is $t = 8.09$ y. You will notice something quite peculiar about the data shown above. First, when the comet crosses the $-x$ axis, its $x$-velocity did not change signs but stayed negative. In fact, by looking through the data, we can find the time interval when the $x$-velocity changes signs as shown below.
Table 3: Position and velocity data for the comet near aphelion, using the velocity as indicator.

<table>
<thead>
<tr>
<th>time (s)</th>
<th>position (m)</th>
<th>velocity (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>276220800.0</td>
<td>$&lt;-1.94433e+12, -5.13384e+10, 0&gt;$</td>
<td>$&lt;-0.174158, -2441.26, 0&gt;$</td>
</tr>
<tr>
<td>276307200.0</td>
<td>$&lt;-1.94433e+12, -5.15493e+10, 0&gt;$</td>
<td>$&lt;2.88516, -2441.18, 0&gt;$</td>
</tr>
</tbody>
</table>

During this interval, the x-velocity goes from a negative value to a positive value. Thus, using x-velocity as an indicator, aphelion occurs at $t = 276220800.0$ s = 8.76 y. Also, during this interval, the x-position is most negative, as expected. The time for the comet to reach aphelion should be half the period, and $17.6 / 2 = 8.8$ y which is consistent. This suggests that we should use the x-velocity as an indicator of aphelion rather than the y-position which seems to be much more uncertain. But either way, we see that the x-position, to three significant figures, is $-1.94e+12$ m. Rounding to two significant figures, the comet’s aphelion distance from Sun is $1.9 \times 10^{12}$ m, since we can only be accurate to two significant figures anyway. Regardless of some peculiarity in the data, we come to the same conclusion for the aphelion distance of the comet from Sun, to two significant figures.

Because of our approximation that $\vec{v}_{avg} = \frac{\vec{p}}{\Delta t}$ and because numerical errors add up (and we are adding error during thousands of time steps), then our answers will have numerical error. But to only two significant figures, the simulation gives fairly good estimates of the comet’s actual period and actual aphelion distance.