Question (15a0001)

Initial speed of a space probe traveling from Earth to Jupiter along a “least energy” orbit

To send a probe from earth to another outer planet, it is most efficient (i.e. conserves the most fuel) to put the probe into an orbit about Sun so that when it is nearest Sun (perihelion), its path is tangent to the earth’s orbit, and when it is furthest from Sun (aphelion), its path is tangent to the outer planet’s orbit. Besides the fuel necessary to put the probe into orbit (i.e. escape from Earth) and to make it orbit the outer planet once it gets there, no fuel is necessary during travel from Earth to the outer planet. For this reason, the probe’s orbit in this case is called a “least-energy" orbit.

Suppose the probe travels from Earth to Jupiter as shown below. The radius of the Earth’s nearly circular orbit is \(150 \times 10^6\) km, and the radius of Jupiter’s nearly circular orbit is \(778 \times 10^6\) km.

![Figure 1: The path of a probe leaving Earth at perihelion and arriving at Jupiter at aphelion.](image)

If the probe’s speed at aphelion should be the same as the speed of Jupiter (\(1.31 \times 10^4\) m/s), what should be the probe’s speed at perihelion, when it leaves Earth with its thrusters turned off? Neglect interactions of the probe with Earth and Jupiter, and assume that the probe’s motion is completely determined by its interaction with Sun.

Solution

Apply the Energy Principle. Define the system to be the space probe and Sun.

\[
\Delta E_{sys} = W_{surr}
\]

Neglect interactions of the system with its surroundings (especially Earth and Jupiter). Thus, there is no work done on the system by the surroundings, and the energy of the system is constant.

\[
\Delta E_{sys} = 0 \\
E_{sys,i} = E_{sys,f}
\]

The energy of the system includes the particle energies of the space probe and Sun and their interaction energy (gravitational potential energy). Define the initial state of the system to be when the space probe is at perihelion (near Earth) and the final state of the system to be when the space probe is at aphelion (near Jupiter). Thus, applying the energy principle...
\[ E_{\text{sys},i} = E_{\text{sys},f} \]
\[ E_{\text{probe},i} + E_{\text{sun},i} + U_{\text{grav},\text{probe},\text{sun},i} = E_{\text{probe},f} + E_{\text{sun},f} + U_{\text{grav},\text{probe},\text{sun},f} \]

Substitute expressions for the particle energies in terms of rest energy and kinetic energy.

\[ E_{\text{rest},\text{probe},i} + E_{\text{rest},\text{sun},i} + K_{\text{probe},i} + K_{\text{sun},i} + U_{\text{grav},\text{probe},\text{sun},i} = E_{\text{rest},\text{probe},f} + E_{\text{rest},\text{sun},f} + K_{\text{probe},f} + K_{\text{sun},f} + U_{\text{grav},\text{probe},\text{sun},f} \]

The particle energy includes both rest energy and kinetic energy. But the final rest energy of the system is the same as the initial rest energy of the system (assuming that neither the probe nor Sun loses or gains significant mass during this process). Therefore, the rest energy cancels. Also, the kinetic energy of Sun is negligible. It’s so massive compared to the probe (and the planets) for that matter that it moves very slightly in its own orbit. This is called a wobble, but is negligible.

\[ K_{\text{probe},i} + U_{\text{grav},\text{probe},\text{sun},i} = K_{\text{probe},f} + U_{\text{grav},\text{probe},\text{sun},f} \]

Solve for the initial kinetic energy. Substitute the known values. Note that the final distance of the probe from Sun is at the orbital radius of Jupiter and the initial position of the probe from Sun is at the orbital radius of Earth.

\[
K_{\text{probe},i} = K_{\text{probe},f} + U_{\text{grav},\text{probe},\text{sun},f} - U_{\text{grav},\text{probe},\text{sun},i} \\
= \frac{1}{2}mv_i^2 + \frac{-GmM_Sun}{r_f} - \frac{-GmM_Sun}{r_i} \\
= \frac{1}{2}mv_i^2 + \frac{-GmM_Sun}{r_f} - \frac{-GmM_Sun}{r_i}
\]

The mass of the space probe \( m \) is not known; therefore, we cannot get a numerical answer for the initial kinetic energy. But we can still solve for the initial speed by substituting an expression for the initial kinetic energy.

\[
\frac{1}{2}mv_i^2 = \frac{1}{2}mv_f^2 + \frac{-GmM_Sun}{r_f} - \frac{-GmM_Sun}{r_i}
\]

Divide both sides by \( m \) and solve for the final speed of the probe.
\[ \frac{1}{2}mv_i^2 = \frac{1}{2}mv_f^2 + \frac{-GmM_{\text{Sun}}}{r_f} - \frac{-GmM_{\text{Sun}}}{r_i} \]
\[ \frac{1}{2}p_i v_i^2 = \frac{1}{2}p_f v_f^2 + \frac{-GpM_{\text{Sun}}}{r_f} - \frac{-GpM_{\text{Sun}}}{r_i} \]
\[ \frac{1}{2}v_i^2 = \frac{1}{2}v_f^2 + \frac{-GM_{\text{Sun}}}{r_f} - \frac{-GM_{\text{Sun}}}{r_i} \]
\[ v_i = \sqrt{2\left(\frac{1}{2}v_f^2 + \frac{-GM_{\text{Sun}}}{r_f} - \frac{-GM_{\text{Sun}}}{r_i}\right)} \]
\[ v_i = \sqrt{\left(\frac{v_f^2}{2} + \frac{-2GM_{\text{Sun}}}{r_f} - \frac{-2GM_{\text{Sun}}}{r_i}\right)} \]

Substitute \( G = 6.7 \times 10^{-11} \text{ Nm}^2/\text{kg}^2 \), \( r_i = 1.5 \times 10^{11} \text{ m} \), \( r_f = 7.78 \times 10^{11} \text{ m} \), \( v_f = 1.31 \times 10^4 \text{ m/s} \), and \( M_{\text{Sun}} = 1.99 \times 10^{30} \text{ kg} \). Then,

\[ v_i = 4.0 \times 10^4 \text{ m/s} \]

The initial speed of the probe when leaving Earth \((4.0 \times 10^4 \text{ m/s})\) is greater than its speed when it arrives at Jupiter \((1.31 \times 10^4 \text{ m/s})\). This makes sense because as it travels to Jupiter, the system gains potential energy and therefore loses kinetic energy.